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SATURATION OF THE CLOSED UNBOUNDED FILTER ON THE SET OF REGULAR CARDINALS

BY

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ABSTRACT. For any $\alpha < \kappa^+$, the following are equiconsistent:
(a) κ is measurable of order α,
(b) κ is α-Mahlo and the filter $\mathbf{C}[\text{Reg}]$ is saturated.

Introduction. Let κ be a regular uncountable cardinal. $\mathbf{C}$ is the filter over κ generated by closed unbounded subsets of κ. If $S \subseteq \kappa$ is a stationary set then

$$\mathbf{C}[S] = \{C \cap S : C \in \mathbf{C}\}$$

is the restriction of $\mathbf{C}$ to S.

By a result of Solovay [**9**], the filter $\mathbf{C}[S]$ is not κ-saturated for any stationary set S; i.e. every stationary set is the union of κ disjoint stationary subsets. A natural question is whether the filter $\mathbf{C}$ can be κ^+-saturated, or more generally, whether $\mathbf{C}[S]$ can be κ^+-saturated for some stationary set S.

The filter $\mathbf{C}[S]$ is κ^+-saturated if there exists no collection of κ^+ stationary subsets of S such that any two of them have nonstationary intersection. To simplify the terminology, we call $\mathbf{C}[S]$ *saturated* instead of κ^+-saturated.

If $\mathbf{C}$ is saturated then κ is a measurable cardinal in an inner model [**3**]; moreover, if κ is a successor cardinal then considerably stronger properties of κ can be deduced [**4, 7**]. In fact, if κ is a successor cardinal and $\kappa > \aleph_1$, then $\mathbf{C}$ itself cannot be saturated [**8**].

If κ is an inaccessible cardinal, let Sing and Reg denote, respectively, the set of all singular and regular cardinals $\alpha < \kappa$. It is not known whether $\mathbf{C}[\text{Sing}]$ can be saturated, it is only known that it is then necessary for κ to be a measurable cardinal of order κ in an inner model. The present paper concerns itself with the saturation of the filter $\mathbf{C}[\text{Reg}]$.

As we assume that the set Reg is stationary, κ is a Mahlo cardinal. We show that there is a deep relationship between the hierarchy of Mahlo cardinals on one hand, and the hierarchy [**6**] of measurable cardinals on the other.

Let S and T be stationary subsets of κ. Following [**2**], we let

$$S < T \quad \text{iff} \quad \text{for almost all } \alpha \in T, S \cap \alpha \text{ is a stationary subset of } \alpha.$$

"For almost all α" means modulo the filter $\mathbf{C}$. The relation $<$ is a well founded partial ordering. The *order* of a stationary set is its rank in $<$; the cardinal κ is *α-Mahlo* if the length of $<$ restricted to subsets of Reg is at least α. Hence 0-Mahlo means inaccessible, 1-Mahlo means Mahlo, 2-Mahlo means that the set of all Mahlo cardinals $< \kappa$ is stationary etc.

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Let κ be a measurable cardinal, and let U and V be normal measures on κ. Following [**6**], we let

$$U \lhd V \quad \text{iff} \quad U \text{ belongs to the ultrapower by } V.$$

The relation $\lhd$ is a well-founded partial ordering. The *order* of U is its rank in $\lhd$; κ is *α-measurable* if the length of $\lhd$ is at least α. Hence 1-measurable means measurable, 2-measurable means that κ has a normal measure such that the set of all measurable cardinals $< \kappa$ has measure one, etc.

If κ is κ^+-Mahlo it is called *greatly Mahlo* in [**1**]. In that case, there is a natural decomposition of κ into κ^+ almost disjoint stationary sets. Consequently, the filter $\mathbf{C}[\text{Reg}]$ is not saturated.

In this paper we solve the problem of saturation of the filter $\mathbf{C}[\text{Reg}]$ in the case when κ is α-Mahlo and $\alpha < \kappa^+$. We prove equiconsistency of "κ is α-Mahlo and $\mathbf{C}[\text{Reg}]$ is saturated" with "κ is α-measurable". More precisely,

THEOREM A. *Let κ be an α-Mahlo cardinal and $0 < \alpha < \kappa^+$. If the filter $\mathbf{C}[\text{Reg}]$ over κ is saturated then κ is α-measurable in an inner model.*

THEOREM B. *Let κ be a measurable cardinal and $0 < \alpha < \kappa^+$. If κ is α-measurable then there is a Boolean-valued model V^B in which κ is an α-Mahlo cardinal and the filter $\mathbf{C}[\text{Reg}]$ over κ is saturated.*

The proof of Theorem A is implicit in [**2**] and runs as follows: We assume that κ is not κ^+-measurable in any inner model and let $\mathbf{K}$ be the core model for measures [**7**]. We claim that κ is α-measurable in $\mathbf{K}$. As in Lemma 4.7 of [**2**], there is a decomposition W of the set Reg into a maximal almost disjoint collection of $\leq \kappa$ stationary subsets S such that, for each of them, $U_S = \mathbf{C}[S] \cap \mathbf{K}$ is a normal measure in the model $\mathbf{K}$. Moreover, the partition W can be obtained such that the length of $<$ restricted to W has length at least α. It follows from [**2**] that if S, $T \in W$ and $S < T$, then $\mathbf{K} \models U_S \lhd U_T$. Hence κ is α-measurable in $\mathbf{K}$.

We devote the rest of our paper to the proof of Theorem B.

Sketch of the forcing construction. We start with a model of ZFC + GCH (the GCH can be made true by a preliminary forcing extension that preserves measurable cardinals and their orders). The idea is to change α-measurable cardinals into α-Mahlo cardinals, and at the same time, change a measure of order β into a saturated filter that coincides with $\mathbf{C}[E_\beta]$ where E_β is the set of all β-Mahlo cardinals. And all that while preserving all cofinalities, as well as GCH. In particular, inaccessible cardinals remain inaccessible; nonmeasurable cardinals become non-Mahlo, measurable cardinals of order 1 become Mahlo cardinals, etc.

If κ is nonmeasurable in the ground model then the set Reg will become nonstationary; in other words, the set Sing will acquire a closed unbounded subset. If κ is measurable of order 1 (in the ground model), we choose a normal measure U on κ. We arrange things so that each set of measure one will remain stationary in the extension. The set Reg is the union of two sets, $E_0 =$ the set of all nonmeasurable cardinals and $E_1 =$ the set of all measurable cardinals, and $E_0 \in U$. We make the set E_1 nonstationary. In the process, for every set A of measure one we make sure that $A \cup \text{Sing}$ acquires a closed unbounded subset. Hence $\mathbf{C}[\text{Reg}]$ will extend U. And we also make sure that $\mathbf{C}[\text{Reg}]$ will be saturated.

If κ is measurable of order 2 then it has a measure U_1 that concentrates on measurable cardinals of order 1. We have Reg $= E_0 \cup E_1 \cup E_2$ where E_0 is the set of all nonmeasurables, E_1 is the set of all measurables of order 1, and $E_2 =$ all measurables of higher order. Now the idea is to keep E_0 and E_1 stationary (while making E_2 nonstationary) and make both $\mathbf{C}[E_0]$ and $\mathbf{C}[E_1]$ saturated; moreover, all elements of E_1 will be Mahlo cardinals (formerly being measurable) and so κ will be a 2-Mahlo cardinal.

The filter $\mathbf{C}[E_1]$ will extend the measure U_1 of the ground model. Thus for every $A \in U_1$, $A \cup (\mathrm{Sing} \cup E_0)$ acquires a closed unbounded subset. In order to deal with E_0, we use a measure U_0 that concentrates on E_0, and eventually extend U_0 to $\mathbf{C}[E_0]$. But we do not choose U_0 arbitrarily; for various reasons U_0 has to cohere with other steps of our constructions and so U_0 is the ultrapower by U_1 of the measures previously chosen on measurable cardinals below κ.

The basic technique used in our construction involves shooting a closed unbounded set through a given stationary set A; our construction is an iteration of this basic technique. The technique is well known and understood: forcing conditions are closed initial segments of the intended closed unbounded set. When the stationary set S contains the set Sing then this notion of forcing does not add bounded subsets of κ (see Lemma 1) and that makes the iteration easier to handle.

Our construction is an iteration with Easton support ("backward Easton"). At stage κ (where κ is an inaccessible cardinal) we perform an iteration Q of length κ^+, with support of size $< \kappa$; we shoot closedc unbounded sets through various stationary subsets of κ.

If κ is nonmeasurable, the iteration Q amounts to nothing more than shooting a club through the set Sing, and then adding κ^+ Cohen subsets of κ. That makes κ a non-Mahlo inaccessible cardinal.

If κ is measurable of order 1, we choose a measure U_0 and iterate shooting a club through sets of the form $A \cup \mathrm{Sing}$ where $A \subseteq E_0$ is stationary. The intention is to make the filter $\mathbf{C}[E_0]$ saturated. To achieve that we employ a variation of the Kunen–Paris method [**5**, **4**]. Let j be the elementary embedding given by U_0. As we perform the iteration, we manage to keep extending j so that the extended embedding is an elementary embedding of the generic extension $V[G]$. This extended embedding j does not live in the generic extension itself, but in a larger extension. However, that larger extension is a κ^+-c.c. extension of $V[G]$ and so j naturally defines a saturated filter over κ. The construction is so arranged that this filter coincides with $\mathbf{C}[E_0]$.

When κ is measurable of higher order, the construction is a more or less natural generalization of the construction outlined above, except that care has to be taken for the various iterations to cohere.

The forcing construction—preliminaries. The basic technique we use is the method of shooting a *club* (a closed unbounded subset) through a given stationary set. Thus let κ be an inaccessible cardinal, and let $S \subseteq \kappa$ be a stationary set. We define a notion of forcing $\mathrm{CU}(S)$ as follows:

$$\mathrm{CU}(S) = \{p : p \subseteq S, |p| < \kappa \text{ and } p \text{ is a closed set of ordinals}\}.$$

A condition p is stronger than q, $p < q$, if q is an initial segment of p. A generic filter on $\mathrm{CU}(S)$ yields a closed unbounded set C that is included in S (thus making the complement of S nonstationary).

In general, there is no reason why cardinals or cofinalities should be preserved by this forcing. If, however, $S \supseteq \mathrm{Sing}$, then the forcing is well behaved:

LEMMA 1. *If $S \supseteq \mathrm{Sing}$ then for every regular $\lambda < \kappa$ there is a dense subset $P \subseteq \mathrm{CU}(S)$ that is λ-closed.*

PROOF. $P = \{p: \sup(p) > \lambda\}$. □

It follows that $\mathrm{CU}(S)$ does not add new bounded subsets of κ (nor new ordinal sequences of length less than κ) and κ remains an inaccessible cardinal. The size of $\mathrm{CU}(S)$ is κ, and so cardinals above κ are also preserved, as is the GCH.

We shall repeatedly apply the forcing notion $\mathrm{CU}(S)$ as follows: Let E be a subset of the set Reg; for instance, $E = E_0 =$ the set of all regular nonmeasurable cardinals, or $E = E_\alpha =$ the set of all measurable cardinals of order α. Let X be a subset of E. We let

$$\begin{aligned}\mathrm{CU}(E, X) &= \{p \subseteq \kappa: |p| < \kappa, p \text{ is closed and } p \cap E \subseteq X\}\\ &= \mathrm{CU}(X \cup (\kappa - E)).\end{aligned}$$

As $X \cup (\kappa - E) \supseteq \mathrm{Sing}$, Lemma 1 applies. The club C obtained by this forcing is such that $C \cap E \subseteq X$, i.e. it avoids the set $E - X$. If X is nonstationary then $\mathrm{CU}(E, X)$ makes the set E nonstationary.

Now consider iterations of $\mathrm{CU}(E, X)$. That is, consider an iteration Q of length ϑ with support of size $< \kappa$, where at stage ξ we force with $\mathrm{CU}(E, X)$ where $X \in V^{Q \restriction \xi}$ (while $E \in V$). We shall describe a uniform representation of such iterations.

Let C_ϑ be the set of all p of the following form:

(i) $p \subset \vartheta \times \kappa$,

(ii) $|p| < \kappa$,

(iii) $p_i = \{\nu < \kappa: (i, \nu) \in p\}$,
$\mathrm{supp}(p) = \{i < \vartheta: p_i \neq \emptyset\}$,
p_i is a closed set of ordinals $(i < \vartheta)$,

(iv) $p < q$ iff $p \supseteq q$ and each q_i is an initial segment of p_i.

LEMMA 2. *If Q is an iteration of length ϑ as above, then*

(a) *for every $\lambda < \kappa$, Q has a dense subset that is λ-closed,*

(b) *Q has a dense subset Q' isomorphic to a subset of C_ϑ, and if p, $q \in Q'$ then $p \cup q \in Q'$.*

PROOF. By simultaneous induction on the length of iteration. □

Thus we identify Q with a subset of C_ϑ.

It is clear from the definition that the partial ordering C_ϑ satisfies the κ^+-chain condition. As Q is a sublattice of C_ϑ, Q does too, and we state this as

LEMMA 3. *Q satisfies the κ^+-chain condition.* □

Let $X \in V^Q$ be a name for a subset of E. For each $\nu \in E$, let W_ν be a maximal set of mutually incompatible conditions $p \in Q$ such that $p \Vdash \nu \in X$. As Q has the κ^+-c.c., W_ν has size at most κ. Let $W \subseteq E \times C_\vartheta$ be as follows:

$$W = \{(\nu, p): \nu \in E, p \in W_\nu\}.$$

We have $|W| \leq \kappa$; let

$$\operatorname{supp} W = \bigcup\{\operatorname{supp}(p): p \in W_\nu, \nu \in E\}.$$

All our iterations will be of length $\leq \kappa^+$. In view of the preceding remark, we have a canonical way of describing subsets of κ at all stages of all iterations:

DEFINITION. A *canonical name* for a subset of $E \subseteq \mathrm{Reg}$ is a set $W \subseteq E \times C_{\kappa^+}$ of size $\leq \kappa$.

Given an iteration Q of length $\leq \kappa^+$ and a generic filter G on Q, the G-interpretation of a canonical name W is

$$\{\nu \in E: (\nu, p) \in W \text{ for some } p \in G\}.$$

In other words, W represents the name $X \in V^Q$ such that

$$||\nu \in X||_Q = \sum\{p \in Q: (\nu, p) \in W\} \qquad (\nu \in E).$$

(No matter that not all p involved are conditions in Q.)

Now let $\{W_\xi: \xi < \kappa^+\}$ be a sequence of canonical names. We associate with this sequence the following iteration Q of length κ^+: At stage ξ of the iteration, let $X \in V^{Q \restriction \xi}$ be the name represented by the canonical name W_ξ (and E the corresponding subset of Reg in the ground model). The ξth stage of the iteration is the notion of forcing $\mathrm{CU}(E, X) \in V^{Q \restriction \xi}$. (Of course, we identify each $Q \restriction \xi$ with a subset of C_{κ^+}.)

Still considering the iteration associated with $\{W_\xi: \xi < \kappa^+\}$, let A be a subset of κ^+ such that

$$(*) \qquad \operatorname{supp} W_\xi \subseteq A \quad \text{whenever} \xi \in A.$$

Let

$$Q_A = \{p \in Q: \operatorname{supp}(q) \subseteq A\}.$$

Using $(*)$ we can see that Q_A is an iteration (indexed by A) associated with $\{W_\xi: \xi \in A\}$, and r.o.(Q_A) (the complete Boolean algebra corresponding to Q_A) is a regular subalgebra of r.o.(Q).

Also if $B \subseteq A$ also satisfies $(*)$, then Q_B is a regular subalgebra of Q_A.

The iterations. We shall now describe the forcing construction in detail in the case $\alpha = 2$. The general case can be obtained by a suitable modification.

Thus we assume that the ground model V satisfies GCH, and that κ is a measurable cardinal of order 2. Let

$$E_0 = \text{ the set of all inaccessible nonmeasurable cardinals below } \kappa,$$
$$E_1 = \text{ the set of all measurable cardinals below } \kappa.$$

Let U_1 be a normal measure on κ concentrating on E_1. For each $\delta \in E_1$ let U_δ be a normal measure on δ concentrating on $E_0 \cap \delta$, and let U_0 be the measure on κ represented in the U_1-ultrapower by $\langle U_\alpha: \alpha < \kappa\rangle$.

The notion of forcing the P we are to construct is an iteration of length $\kappa + 1$, with Easton support. That is, we take direct limits at regular stages, and inverse limits otherwise.

All cardinals and cofinalities are preserved at each step of this construction, as is the GCH.

A nontrivial iteration occurs only at inaccessible stages; when δ is not inaccessible than $P_{\delta+1} = P_\delta$. The final notion of forcing is $P = P_{\kappa+1}$.

When δ is inaccessible, then $P_{\delta+1} = P_\delta * Q_\delta$ where Q_δ is, in V^{P_δ}, a notion of forcing of the form Q_A, where Q the iteration (of length δ^+ with $< \delta$ support) associated with some δ^+-sequence of canonical terms, and A is a suitable subset of δ^+ that satisfies $(*)$.

We define $Q_\delta \in V^P$ by induction on δ, depending on whether $\delta \in E_0$, $\delta \in E_1$ or $\delta = \kappa$. In each case we use a certain δ^+-sequence of canonical names $\langle W_\xi \colon \xi < \delta^+ \rangle$. Each sequence has these properties:

(i) if ξ is even then W_ξ is a name for a subset of $E_0 \cap \delta$; if ξ is odd then W_ξ is a name for a subset of $E_1 \cap \delta$,

(ii) the sequence enumerates all canonical names of subsets of $E_0 \cap \delta$ and $E_1 \cap \delta$; moreover,

(iii) each canonical name appears δ^+ times.

Case I: $\delta \in E_0$. In this case (δ not measurable) we choose any δ^+-sequence of canonical names with properties (i), (ii), (iii) and let $Q_\delta = Q$ be the iteration associated with the sequence.

Note that because both $\mathrm{CU}(E_0, \emptyset)$ and $\mathrm{CU}(E_1, \emptyset)$ appear in the iteration, both E_0 and E_1 become nonstationary and hence the set of all singular cardinals below δ has a closed unbounded subset in $V^{P_{\delta+1}}$. The result is that δ, while it remains inaccessible, is a non-Mahlo cardinal in $V^{P_{\delta+1}}$ (and therefore in V^P).

We note in passing that the notion of forcing Q is the same as a two step iteration, first shooting a closed unbounded set through Sing and then adding δ^+ Cohen subsets of δ.

Case II: $\delta \in E_1$. We have chosen a normal measure U_δ on δ that concentrates on $E_0 \cap \delta$. Let j denote the elementary embedding $j \colon V \to M$ where M = the ultrapower of V by U_δ. In the model M, $j(P_\delta)$ is an iteration of length $j(\delta)$, with Easton support, and can be factored out as

$$j(P_\delta) = P_\delta * \tilde{Q}_\delta * R$$

where $\tilde{Q}_\delta \in M^P$ is the forcing done at stage δ, and R has a dense subset that is δ-closed. In M, δ is a nonmeasurable cardinal (because $E_0 \in U_\delta$) and so $\tilde{Q}_\delta$ is, in M^{P_δ}, the iteration of length δ^+ associated with some δ^+-sequence $\langle W_\xi \colon \xi < \delta^+ \rangle$ of canonical names.

Because M is an ultrapower of V and because $|P_\delta| = \delta$, it follows that M^{P_δ} has the same ordinal δ-sequences as V^{P_δ}. Hence $\langle W_\xi \colon \xi < \delta^+ \rangle$ is a sequence of canonical names in V^{P_δ} satisfying (i), (ii), (iii), and $Q = \tilde{Q}_\delta$ is the associated iteration of length δ^+ (in V^{P_δ}). Thus we have defined Q.

We shall now define $A \subseteq \delta^+$. We work inside the model $V_1 = V^{P_\delta}$ (or, equivalently, we choose some generic filter $G_1 \subset P$ and let $V_1 = V[G_1]$).

LEMMA 4. *The elementary embedding $j \colon V \to M$ can be extended to an elementary embedding $j \colon V_1 \to M_1$ that lives in V_1^Q.*

PROOF. It is enough to find in V_1^Q a filter $H_1 \subseteq R$ generic over $M^{P_\delta * Q}$. Then, if $G_1 \subset P$ is V-generic and $G \subset Q$ is $V[G_1]$-generic, we let $j(G_1) = G_1 \times G \times H_1$ and extend j to $V_1 = V[G_1]$ accordingly. The forcing notion R has a dense subset that is δ-closed. Moreover, the size of the power set of R (in $M^{P_\delta * Q}$) is $j(\delta^+)$, and

$|j(\delta^+)| = \kappa^+$ (in V). Thus one can find in V_1^Q an M^{P*Q}-generic filter H_1, and consequently extend j to an elementary embedding $j: V_1 \to M_1$ where

$$M_1 = M^{P_\delta * Q}[H_1]. \quad \square$$

We construct $A \subseteq \delta^+$ by induction, that is we construct $A \cap \alpha$ for $\alpha < \delta^+$. Simultaneously, we choose on the side V_1^Q-names q_α for conditions in $j(Q_{A\cap\alpha})$. The purpose of these conditions is to produce, in V_1^Q, an M_1-generic filter H on $j(Q_A)$, thus making it possible to further extend j to an elementary embedding $j: V_1^{Q_A} \to M_1[H]$.

The conditions q_α, $\alpha < \delta^+$, will form a descending chain. In order for the induction to continue at limit stages, we require that, for each $\xi \in \operatorname{supp}(q_\alpha)$, the supremum of the ξth coordinate of q is at least δ.

In V_1^Q, let G be the canonical generic filter on Q; G yields a sequence $\langle C_\xi : \xi < \delta \rangle$ of closed unbounded subsets of δ. If q is a condition in $j(Q)$ and $\eta < j(\delta)$ then the ηth coordinate of q is a closed bounded subset of $j(\delta)$. We make a further requirement for each $\alpha < \delta^+$:

(**) For each $\xi \in A \cap \alpha$, the $j(\xi)$th coordinate of q_α extends C_ξ.

Note that (**) entails that δ is a member of the $j(\xi)$th coordinate of q_α.

In order to have the q_α produce an M_1-generic filter, we choose an enumeration $\langle a_\alpha : \alpha < \delta^+ \rangle$ of all V_1^Q-names for an antichain of $j(Q)$ in M_1, such that each a_α occurs δ^+ many times. This is possible because of the size of $j(Q)$ and because V_1^Q is a κ^+-c.c. extension of V_1.

Suppose that $A \cap \xi$ and q_ξ have been constructed for all $\xi < \alpha$. If α is a limit than $A \cap \alpha = \bigcup_{\xi<\alpha}(A \cap \xi)$ and we find a Q-name for a condition q_α in $j(Q_{A\cap\alpha})$ below the q_ξ's such that q_α is below a_α if possible (a_α may not be maximal, or its elements may not all be in $j(Q_{A\cap\alpha})$). We also make sure that q_α satisfies the inductive requirement on the supremums (the requirement (**) is satisfied automatically).

Now suppose that $A \cap \alpha$ and q_α have been defined; we are to find q_α and to decide whether $\alpha \in A$. Here we distinguish between the case when α is even and the case when α is odd. However, in either case we do not let $\alpha \in A$ unless $\operatorname{supp}(W_\alpha) \subseteq A \cap \alpha$, where W_α is the αth canonical name in the definition of Q. This is to insure that A satisfies (*). If $\operatorname{supp}(W_\alpha) \not\subseteq A \cap \alpha$ then we let $A \cap (\alpha+1) = A \cap \alpha$, and choose a Q-name for $q_{\alpha+1} \in j(Q_{A\cap\alpha})$ below q_α such that $q_{\alpha+1}$ is below $a_{\alpha+1}$ if possible (and satisfies the inductive requirements).

If $\operatorname{supp}(W_\alpha) \subseteq A \cap \alpha$ then we distinguish between α odd and α even. We recall that $E_0 \in U_0$ and therefore $\delta \in j(E_0)$, $\delta \notin j(E_1)$.

Case A: *α odd.* We let $\alpha \in A$. This means that we let Q_A shoot a club through $W_\alpha \cup (\kappa - E_1)$. As we do this for all odd α, we particularly include $\mathrm{CU}(E_1, \varnothing)$ in the iteration and therefore make $E_1 \cap \delta$ nonstationary.

We also choose a Q-name $q_{\alpha+1} \in j(Q_{A\cap(\alpha+1)})$ below q_α such that $q_{\alpha+1}$ is below $a_{\alpha+1}$ if possible and satisfies the inductive requirements. In order to satisfy (**) it is necessary that $C_\alpha \cup \{\delta\}$ is the $j(\alpha)$th coordinate of a condition in $j(Q)$. In other words, $C_\alpha \cup \{\delta\}$ has to be a condition in $\mathrm{CU}(j(E_1 \cap \delta), j(W_\alpha))$. But every initial segment of C_α is a condition in $\mathrm{CU}(E_1, W_\alpha)$ because it is the αth coordinate of a condition in Q, and so the only question is whether $C_\alpha \cup \{\delta\} \subseteq j(W_\alpha \cup (\delta - E_1))$. But, as we remarked, $\delta \notin j(E_1)$, and so it is possible to satisfy (**).

Case B: α *even.* First we choose a Q-name q for a condition in $j(Q_{A\cap\alpha})$ below q_α such that, in V_1^Q, q decides $\delta \in j(W_\alpha)$. That is, we have

$$[\![\text{either } q \Vdash \delta \in j(W_\alpha), \text{ or } q \Vdash \delta \notin j(W_\alpha)]\!]_Q = 1.$$

If $[\![q \Vdash \delta \in j(W_\alpha)]\!]_Q \neq 1$, then we let $A \cap (\alpha+1) = A \cap \alpha$, and choose $q_{\alpha+1}$ below q, below $a_{\alpha+1}$ if possible, subject to the inductive requirement. If $[\![q \Vdash \delta \in j(W_\alpha)]\!]_Q = 1$, then we let $\alpha \in A$, and fine $q_{\alpha+1} \in j(Q_{A\cap(\alpha+1)})$ below q, below $a_{\alpha+1}$ if possible, and such that $q_{\alpha+1}$ satisfies the inductive requirements.

Arguing in V_1^Q, we use the fact that $q \Vdash \delta \in j(W_\alpha)$. Hence $q \in M_1^{j(Q_{A\cap\alpha})}$ forces

$$C_\alpha \cup \{\delta\} \in \mathrm{CU}(j(E_0 \cap \delta), j(W_\alpha))$$

and hence there is a condition in $j(Q_{A\cap(\alpha+1)})$ that extends q and whose $j(\alpha)$th coordinate extends C_α. Thus a $q_{\alpha+1}$ can be found that satisfies $(**)$.

This completes the construction of $A \subseteq \delta^+$, and we let $Q_\delta = Q_A$, and $P_{\delta+1} = P_\delta * Q$.

Since every maximal antichain of $j(Q_A)$ appears in the sequence $\langle a_\alpha : \alpha < \delta^+\rangle$, the conditions q yield (in V_1^Q) an M_1-generic filter H on $j(Q_A)$. Moreover, for each $\alpha \in A$ the club $C_\alpha \subseteq \delta$ is an initial segment of the $j(\alpha)$th coordinate of H. Thus if we let $j(G \restriction A) = H$, we get an extension of the elementary embedding $j: V_1 \to M_1$ to an elementary embedding

$$j: V_1[G \restriction A] \to M_1[H],$$

in other words to an elementary embedding j of $V_1^{Q_A}$ (and j lives in V_1^Q).

Let Q/Q_A denote the complete Boolean algebra (in $V_1^{Q_A}$) such that $Q_A * (Q/Q_A) = Q$. As both Q and Q_A satisfy the κ^+-chain condition, Q/Q_A does as well. The elementary embedding j of $V_1^{Q_A}$ in V^Q produces naturally a filter $F \in V_1^{Q_A}$ on δ:

$$x \in F \text{ iff every condition in } Q/Q_A \text{ forces } \delta \in j(X).$$

As Q/Q_A has the κ^+-c.c., F is κ^+-saturated (in $V_1^{Q_A}$). In fact, if we let (in $V_1^{Q_A}$)

$$e(X) = ||\delta \in j(X)||_{Q/Q_A} \qquad (X \subseteq \delta),$$

then e is an embedding of the Boolean algebra $P(\delta)/F$ into Q/Q_A.

Working in $V_1^{Q_A}$, we show that $F = \mathbf{C}[E_0 \cap \delta]$. The definition of F can be restated as follows:

$$\begin{aligned} X \in F \quad &\text{iff } [\![\exists q \in H\ q \Vdash \delta \in j(X)]\!]_Q = 1, \\ &\text{iff } \exists \alpha [\![q_\alpha \Vdash \delta \in j(X)]\!]_Q = 1. \end{aligned}$$

It is clear that every closed unbounded set belongs to F. It is also easy to see that $E_0 \cap \delta \in F$ (because $\delta \in j(E_0 \cap \delta)$). Thus $F \supseteq \mathbf{C}[E_0 \cap \delta]$.

If $X \in F$ then there is a sufficiently large α such that W_α is a name for F, $\mathrm{supp}(W_\alpha) \subseteq A \cap \alpha$, and $[\![q_\alpha \Vdash \delta \in j(X)]\!]_Q = 1$. Then $\alpha \in A$, and C_α is a closed unbounded subset of $X \cup (\delta - E_0)$. Hence there is a club $C_\alpha \in V_1^{Q_A}$ such that $C_\alpha \cap E_0 \subseteq X$, and so $X \in \mathbf{C}[E_0 \cap \delta]$. Therefore, $F = \mathbf{C}[E_0 \cap \delta]$.

Thus $\mathbf{C}[E_0 \cap \delta]$ is saturated and δ is a Mahlo cardinal, in $V^{P_{\delta+1}}$. As further iteration does not add δ-sequences, δ will remain a Mahlo cardinal.

Case III: $\delta = \kappa$. So far, we have constructed P_κ, an iteration of length κ. Let $V_1 = V^{P_\kappa}$ (or $V_1 = V[G_1]$ where G_1 is a generic filter on P_κ). In V_1, both E_0 and E_1 are stationary sets, every element of E_0 is 0-Mahlo (inaccessible and non-Mahlo) and every element of E_1 is 1-Mahlo. We shall construct Q_κ so that in $V_1^{Q_\kappa} = V^{P_\kappa * Q_\kappa} = V^P$, E_0 and E_1 remain stationary, and both $\mathbf{C}[E_0]$ and $\mathbf{C}[E_1]$ are saturated. Thus in V^P, κ is a 2-Mahlo cardinal and $\mathbf{C}[\mathrm{Reg}]$ is saturated.

U_0 and U_1 are the two measures on κ introduced previously, and $j^0: V \to M^0$ and $j^1: V \to M^1$ are the corresponding elementary embeddings.

First we define Q. It is the iteration of closed unbounded forcing of length κ^+, associated with a κ^+-sequence $\langle W_\alpha : \alpha < \kappa^+\rangle$ of canonical names. We let $\langle W_\alpha\rangle$ be the sequence in the ultrapower M^0 determined by the δ^+-sequences we have chosen at stages $\delta \in E_0$. Because $M^0 \subseteq M^1 \subseteq V$ and $|P_\kappa| = \kappa$, the definition of Q from $\langle W_\alpha\rangle$ is the same in each $M^0[G_1]$, $M^1[G_1]$ and $V[G_1]$.

Working in V_1, we now construct a set $A_1 \subseteq \kappa^+$ so as to take $Q_\delta = Q_{A_1}$. In order to satisfy the condition

$$(*) \qquad \mathrm{supp}(W_\alpha) \subseteq A_1 \quad \text{whenever } \alpha \in A_1,$$

we put $\alpha \in A_1$ only if supp $W_\alpha \subseteq A_1 \cap \alpha$. It follows from the way we construct A_1 that the saturation of $\mathbf{C}[E_0]$ is guaranteed by the iteration below κ, and the saturation of $\mathbf{C}[E_1]$ will be achieved by a side construction of a sequence $\{q_\alpha\}$ of names for conditions. We recall that if α is even than W_α is a canonical name for a subset of E_0 (and the αth stage in the iteration Q has the form $\mathrm{CU}(E_0, W_\alpha)$), and if α is odd then W_α is a name for a subset of E_1.

We use the embedding $j^1: V \to M^1$. We have $j^1(P_\kappa) = P_\kappa * Q_{A_0} * R$ where R has a dense κ-closed subset, and $A_0 \in M^1[G_1]$ is a subset of κ^+. As $\kappa \in j^1(E_1)$, the set A_0 is constructed in M^1 as described in the case $\delta \in E_1$. We construct A_1 so that $A_1 \subseteq A_0$.

As in Lemma 4, the elementary embedding $j^1: V \to M^1$ extends to an elementary embedding $j: V_1 \to M_1^1$ that lives in V_1^Q. The embedding is defined by $j(G_1) = G_1 \times (G \restriction A_0) \times H_1$, where H_1 is a $M^1[G_1][G \restriction A_0]$-generic filter on R.

Along with $A_1 \subseteq A_0$ we construct the sequence $\langle q_\alpha : \alpha < \kappa^+\rangle$. Each q_α is a V_1^Q-name for a condition in $j(Q_{A\cap\alpha})$. The conditions form a descending chain; we require that, for each $\xi \in \mathrm{supp}(q_\alpha)$, the supremum of the ξth coordinate of q_α is at least κ. We further require:

$$(**) \qquad \text{For each } \xi \in A \cap \alpha, \text{ the } j(\xi)\text{th coordinate of } q_\alpha \text{ extends } C_\xi.$$

Here C_ξ is the club $\subseteq \kappa$ given by the ξth coordinate of G. We also choose an enumeration $\langle a_\alpha : \alpha < \kappa^+\rangle$ of all V_1^Q-names for antichains of $j(Q)$ in M_1^1 such that each a_α occurs κ^+ many times.

Suppose that $A_1 \cap \xi$ and q_ξ have been defined for all $\xi < \alpha$. If α is a limit then $A_1 \cap \alpha = \bigcup_{\xi<\alpha}(A_1 \cap \xi)$ and we find a Q-name for $q_\alpha \in j(Q_{A_1\cap\alpha})$ below the q_ξ's such that q_α is below a_α if possible, and such that q_α satisfies the inductive requirement.

Now suppose that $A_1 \cap \alpha$ and q_α have been constructed. If $\mathrm{supp}(W_\alpha) \not\subseteq A_1 \cap \alpha$ or if $\alpha \notin A_0$ then we let $A_1 \cap (\alpha + 1) = A_1 \cap \alpha$, and choose $q_{\alpha+1} \in j(Q_{A_1\cap\alpha})$ below q_α, below $a_{\alpha+1}$ if possible, subject to the inductive requirement.

If supp $(W_\alpha) \subseteq A_1 \cap \alpha$ and $\alpha \in A_0$, then there are two cases: α even and α odd.

Case A: α *even.* We let $\alpha \in A_1$. We choose a Q-name $q_{\alpha+1} \in j(Q_{A_1 \cap (\alpha+1)})$ below q_α, below $a_{\alpha+1}$ if possible, such that $q_{\alpha+1}$ satisfies the inductive requirements. Since $\kappa \notin j(E_0)$, it is possible to satisfy (**).

Case B: α *odd.* We choose a Q-name q for a condition in $j(Q_{A_1 \cap \alpha})$ below q_α such that in V_1^Q, q decides $\delta \in j(W_\alpha)$. If $[\![q \Vdash \delta \in j(W_\alpha)]\!]_Q \neq 1$, then we let $A_1 \cap (\alpha+1) = A_1 \cap \alpha$, and choose $q_{\alpha+1}$ below q, below $a_{\alpha+1}$ if possible, subject to the inductive requirement.

If $[\![q \Vdash \delta \in j(W_\alpha)]\!]_Q = 1$, then we let $\alpha \in A_1$, and find $q_{\alpha+1} \in j(Q_{A_1 \cap (\alpha+1)})$ below q, below $a_{\alpha+1}$ if possible, and such that $q_{\alpha+1}$ satisfies the inductive requirements. The same argument we used in the case $\delta \in E_1$, α even, shows that a $q_{\alpha+1}$ can be found that satisfies (**).

This completes the construction of $A_1 \subseteq A_0$, and we let $Q_\kappa = Q_{A_1}$, and $P_\kappa = P_{\kappa+1} = P_\kappa * Q_\kappa$. We have to show that both $\mathbf{C}[E_0]$ and $\mathbf{C}[E_1]$ are saturated in $V^P = V_1^{Q_{A_1}}$.

Saturation of $\mathbf{C}[E_1]$.

The conditions q_α produce (in V_1^Q) an M_1^1-generic filter H on $j(Q_{A_1})$. For each $\alpha \in A_1$, the club $C_\alpha \subseteq \kappa$ is an initial segment of the $j(\alpha)$th coordinate of H and so if we let $j(G \restriction A_1) = H$, we can extend the elementary embedding $j: V_1 \to M_1^1$ to an elementary embedding of $V_1^{Q_{A_1}}$. For $X \subseteq \delta$ in $V_1^{Q_{A_1}}$, let

$$X \in F \text{ iff every condition in } Q/Q_{A_1} \text{ forces } \delta \in j(X).$$

The filter F is κ^+-saturated, and clearly $\mathbf{C}[E_1] \subseteq F$.

The proof that $F = \mathbf{C}[E_1]$ is as in the case of $\mathbf{C}[E_0 \cap \delta]$, but with a little twist. Let $X \in F$. There is a sufficiently large odd α such that W_α is a name for F, $\operatorname{supp}(W_\alpha) \subseteq A_1 \cap \alpha$, and $[\![q_\alpha \Vdash \delta \in j(X)]\!]_Q = 1$. Because of the definition of A_0 in $M_1^{P_\kappa}$ (see the case $\delta \in E_1$, α odd), we have $\alpha \in A_0$. Hence $\alpha \in A_1$, and so there is a club $C_\alpha \in V_1^{Q_{A_1}}$ such that $C_\alpha \cap E_1 \subseteq X$, and so $X \in \mathbf{C}[E_1]$. Therefore, $F = \mathbf{C}[E_1]$.

Saturation of $\mathbf{C}[E_0]$.

We want to show that in $V_1^{Q_{A_1}}$, the filter $\mathbf{C}[E_0]$ is saturated. We shall argue in several steps: one, in $(M^1)^{P_\kappa * Q_{A_0}}$, $\mathbf{C}[E_0]$ is saturated; two, in $V_1^{Q_{A_0}}$, $\mathbf{C}[E_0]$ is saturated; three, in $V^{Q_{A_1}}$, $\mathbf{C}[E_0]$ is saturated. Of course, $\mathbf{C}[E_0]$ is a different filter in each model.

The iteration Q_{A_0} in $(M^1)^{P_\kappa}$ has been constructed so that $\mathbf{C}[E_0]$ is saturated in $(M^1)^{P_\kappa * Q_{A_0}}$; the construction is the U_1-ultrapower of the construction we performed on all $\delta \in E_1$. Because M_1 is an ultrapower of V, and $P_\kappa * Q_{A_0}$ has the κ^+-c.c., the model $(M^1)^{P_\kappa * Q_{A_0}}$ is closed under κ-sequences in $V_1^{Q_{A_0}}$. Thus $\mathbf{C}[E_0]$ is the same filter in $V_1^{Q_{A_0}}$ as in $(M^1)^{P_\kappa * Q_{A_0}}$. In $(M^1)^{P_\kappa * Q_{A_0}}$, we have an embedding e of the algebra $P(\kappa)/\mathbf{C}[E_0]$ into Q/Q_{A_0}. But Q/Q_{A_0} is κ^+-saturated even in $V_1^{Q_{A_0}}$. Hence $\mathbf{C}[E_0]$ is a saturated filter in $V_1^{Q_{A_0}}$.

As $A_1 \subseteq A_0$, we have $V_1^{Q_{A_1}} \subseteq V_1^{Q_{A_0}}$ and there may be stationary sets in $V_1^{Q_{A_1}}$ that are nonstationary in $V_1^{Q_{A_0}}$. However, we prove the following:

LEMMA 5. *We work in $V_1^{Q_{A_1}}$. For every $X \subseteq E_0$, if X is stationary, then*

$$[\![X \text{ is stationary}]\!]_{Q_{A_0}/Q_{A_1}} \neq 0.$$

We give a proof of the lemma below. Granted Lemma 5, we argue as follows, in $V^{Q_{A_1}}$: If the filter $\mathbf{C}[E_0]$ is not saturated, there is an almost disjoint family $\{S_\xi : \xi < \kappa^+\}$ of stationary sets. Because in $V^{Q_{A_0}}$, $\mathbf{C}[E_0]$ is saturated, we have

$$[\![\exists\eta\ \forall\xi \geq \eta\ S_\xi \text{ is nonstationary}]\!]_{Q_{A_0}/Q_{A_1}} = 1.$$

But Q_{A_0}/Q_{A_1} has the κ^+-c.c. and therefore

$$\exists\eta\ \forall\xi \geq \eta[\![S_\xi \text{ is nonstationary}]\!] = 1.$$

And that contradicts Lemma 5.

PROOF OF LEMMA 5. We want to prove that if $X \subseteq E_0$ and if $[\![X$ is nonstationary$]\!] = 1$ then X is nonstationary. Equivalently, work in V_1 and let $X \in V_1^{Q_{A_1}}$ be such that $[\![X \subseteq E_0]\!]_{Q_{A_1}} = 1$. Let σ be the sentence

"there is a club C such that $C \cap E_0 \subseteq X$".

It is enough to show that

$$[\![\sigma]\!]_{Q_{A_0}} = 1 \text{ implies } [\![\sigma]\!]_{Q_{A_1}} = 1.$$

Thus let $X \in V_1^{Q_{A_1}}$ be a name for a subset of E_0. There are arbitrarily large even $\alpha < \kappa^+$ such that the canonical name W_α is a name for X, and supp $W_\alpha \subseteq A_1$. Because $||\sigma||_{Q_{A_0}} = 1$, there is a canonical name C for a club such that $C \in V_1^{Q_{A_0}}$, supp$(C) \subseteq A_0$, and $[\![C \cap E_0 \subseteq X]\!]_{Q_{A_0}} = 1$. Let β be an upper bound for supp(C).

Let $\alpha \geq \beta$ be such that W_α is a name for X and supp$(W_\alpha) \subseteq A_1$. We look at the construction of A_0 in $(M^1)^{P_\kappa}$. First, we have $C \in (M^1)^{P_\kappa * Q_{A_0}}$; since supp$(C) \subseteq \beta \leq \alpha$, we have in fact $C \in (M^1)^{P_\kappa * Q_{A_0 \cap \alpha}}$, and

$$(M^1)^{P_\kappa} \Vdash [\![C \cap E_0 \subseteq W_\alpha]\!]_{Q_{A_0 \cap \alpha}} = 1.$$

According to the definition of A_0 in $(M^1)^{P_\kappa}$ (see the case $\delta \in E_1$, α even), this guarantees that $\alpha \in A_0$. Hence by the definition of A_1 (α even), we have $\alpha \in A_1$, and consequently $[\![\sigma]\!]_{Q_{A_1}} = 1$. $\square$

The general case. The construction of P for a 2-measurable cardinal generalizes to measurable cardinals of higher order. We shall very briefly outline the construction of Q_κ for the case of a measurable cardinal of order ω.

We have normal measures $U_0 \lhd U_1 \lhd U_2 \lhd \cdots \lhd U_n \lhd \cdots$ on κ, concentrating on sets $E_0, E_1, E_2, \ldots, E_n, \ldots$ where E_n is the set of all n-measurable cardinals. Instead of even and odd, we divide ordinals $< \kappa^+$ into ordinals of type n, for $n \in \omega$. We have elementary embeddings $j^n : V \to M^n$ corresponding to the ultrapowers by the U_n. We let $\langle W_\alpha : \alpha < \kappa^+ \rangle$ be the sequence of canonical names (of subsets of the E_n) given by the embedding j^0. Thus $j^0(P_\kappa) = P_\kappa * Q * R_0$, where Q is the iteration associated with $\langle W_\alpha \rangle$. For each n we have

$$j^{n+1}(P_\kappa) = P_\kappa * Q_{A_n} * R_{n+1}$$

where $A_n \in (M^{n+1})^{P_\kappa}$ and $\kappa^+ \supseteq A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n \supseteq \cdots$.

We construct $A = A_\omega$ as follows: If α is an ordinal of type n then we let $a \in A_\omega$ just in case supp$(W_\alpha) \subseteq A_n \cap \alpha$, and $\alpha \in A_n$. Because of the construction of P_κ, if α satisfies these two conditions, we also have $\alpha \in A_{n+1}$, $\alpha \in A_{n+2}$, etc. Hence $A_\omega \subseteq \bigcap_{n=0}^{\infty} A_n$.

In this case when the order of κ is a limit ordinal, we do not have to construct the side conditions. To show that $\mathbf{C}[E_n]$ is saturated in $V_1^{Q_{A_1}}$, we argue as in the case $\delta = \kappa$, saturation of $\mathbf{C}[E_0]$: In $(M^{n+1})^{P_\kappa * Q_{A_n}}$, $\mathbf{C}[E_n]$ is saturated; hence in $(V_1)^{Q_{A_n}}$, $\mathbf{C}[E_n]$ is saturated; hence in $V_1^{Q_{A_\omega}}$, $\mathbf{C}[E_n]$ is saturated.

When κ is a measurable cardinal of a successor order, say $\gamma + 1$, then while constructing the set A_γ we construct on the side V_1^Q-names for conditions in $j^{\gamma+1}(Q_{A\cap\alpha})$, to guarantee saturation of the filter $\mathbf{C}[E_\gamma]$.

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